FOURTH EDITION

Mathematical Proofs

A Transition to Advanced Mathematics

Gary Chartrand
Albert D. Polimeni
Ping Zhang

Fourth Edition

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A Transition to Advanced Mathematics

Gary Chartrand
Western Michigan University

Albert D. Polimeni
State University of New York at Fredonia

Ping Zhang
Western Michigan University

Pearson 330 Hudson Street, NY, NY 10013

Director, Portfolio Management: Deirdre Lynch
Executive Editor: Jeff Weidenaar
Editorial Assistant: Jennifer Snyder
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To

the memory of my mother and father G.C.

the memory of my brother Russ A.D.P.

my mother and the memory of my father P.Z.

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Preface to the Fourth Edition

As the teaching of calculus in many colleges and universities has become more problem-oriented with added emphasis on the use of calculators and computers, the theoretical gap between the material presented in calculus and the mathematical background expected (or at least hoped for) in advanced calculus and other more advanced courses has widened. In an attempt to narrow this gap and to better prepare students for the more abstract mathematics courses to follow, many colleges and universities have introduced courses that are now commonly called "transition courses." In these courses, students are introduced to problems whose solution involves mathematical reasoning and a knowledge of proof techniques, where writing clear proofs is emphasized. Topics such as relations, functions and cardinalities of sets are encountered throughout theoretical mathematics courses. Lastly, transition courses often include theoretical aspects of number theory, combinatorics, abstract algebra and calculus. This textbook has been written for such a course.

The idea for this textbook originated in the early 1980s, long before transition courses became fashionable, during the supervision of undergraduate mathematics research projects. We came to realize that even advanced undergraduates lack a sound understanding of proof techniques and have difficulty writing correct and clear proofs. At that time, we developed a set of notes for these students. This was followed by the introduction of a transition course, for which a more detailed set of notes was written. The first edition of this book emanated from these notes, which in turn has ultimately led to this fourth edition.

While understanding proofs and proof techniques and writing good proofs are major goals here, these are not things that can be accomplished to any great degree in a single course during a single semester. These must continue to be emphasized and practiced in succeeding mathematics courses.

Our Approach

Since this textbook originated from notes that were written exclusively for undergraduates to help them understand proof techniques and to write good proofs, the tone is student-friendly. Numerous examples of proofs are presented in the text. Following common practice, we indicate the end of a proof with the square symbol ■. Often we precede a proof by a discussion, referred to as a *proof strategy*, where we think through what is needed to present a proof of the result in question. Other times, we find it useful to reflect on a proof we have just presented to point out certain key details. We refer to a discussion of this type as a *proof analysis*. Periodically, we present and solve problems, and we may find it convenient to discuss some features of the solution, which we refer to simply as an *analysis*. For clarity, we indicate the end of a discussion of a proof strategy, proof analysis, analysis, or solution of an example with the diamond symbol ♦.

A major goal of this textbook is to help students learn to construct proofs of their own that are not only mathematically correct but clearly written. More advanced mathematics students should strive to present proofs that are convincing, readable, notationally consistent and grammatically correct. A secondary goal is to have students gain sufficient knowledge of and confidence with proofs so that they will recognize, understand and appreciate a proof that is properly written.

As with the first three editions, the fourth edition of this book is intended to assist the student in making the transition to courses that rely more on mathematical proof and reasoning. We envision that students taking a course based on this book have probably had a year of calculus (and possibly another course such as elementary linear algebra) although no specific prerequisite mathematics courses are required. It is likely that, prior to taking this course, a student's training in mathematics consisted primarily of doing patterned problems; that is, students have been taught methods for solving problems, likely including some explanation as to why these methods worked. Students may very well have had exposure to some proofs in earlier courses but, more than likely, were unaware of the logic involved and of the method of proof being used. There may have even been times when the students were not certain what was being proved.

New to This Edition

The following changes and additions to the third edition have resulted in this fourth edition of the text:

- Presentation slides in PDF and LaTeX formats have been created to accompany every chapter. These presentations provide examples and exposition on key topics. Using short URLs (which are called out in the margin using the icon), these presentations are linked in the text beside the supplemental exercises at the end of each chapter. These slides can be used by instructors in lecture, or by students to learn and review key ideas.
- The new Chapter 7, "Reviewing Proof Techniques," summarizes all the techniques that have been presented. The placement of this chapter allows instructors and students to review all of these techniques before beginning to explore different contexts in which proofs are used. This new chapter includes many new examples and exercises.
- The new Chapter 13, "Proofs in Combinatorics," has been added because of a demand for such a chapter. Here, results and examples are presented in this important area of discrete mathematics. Numerous exercises are also included in this chapter.
- The new online Chapter 18, "Proofs with Real and Complex Numbers," has been added to provide information on these two important classes of numbers. This chapter includes many important classical results on real numbers as well as important results from complex variables. In each case, detailed proofs are given.

Chapters 16 (Proofs in Ring Theory), 17 (Proofs in Linear Algebra), and 19 (Proofs in Topology) continue to exist online to allow faculty to tailor the course to meet their specific needs.

- More than 250 exercises have been added. Many of the new exercises fall into the moderate difficulty level and require more thought to solve.
- Section exercises for each chapter have been moved from the end of the chapter to the end of each section within a chapter. (Exceptions: for the review chapter [7] and the online chapters [16–19], all exercises remain at the end of the chapter.)
- In the previous edition, there were exercises at the end of each chapter called Additional Exercises. These summative exercises served to pull together the ideas from the various sections of the chapter. These exercises remain at the end of the chapters in this edition, but they have been renamed Supplemental Exercises.

Contents and Structure

Outline of Contents

Each of the Chapters 1–6 and 8–15 is divided into sections, and exercises for each section occur at the end of that section. There is also a final supplemental section of exercises for the entire chapter appearing at the end of that chapter.

Since writing good proofs requires a certain degree of competence in writing, we have devoted **Chapter 0** to writing mathematics. The emphasis of this chapter is on effective and clear exposition, correct usage of symbols, writing and displaying mathematical expressions, and using key words and phrases. Although every instructor will emphasize writing in his or her own way, we feel that it is useful to read Chapter 0 periodically throughout the course. It will mean more as the student progresses through the course.

Chapter 1 contains a gentle introduction to sets, so that everyone has the same background and is using the same notation as we prepare for what lies ahead. No proofs involving sets occur until Chapter 4. Much of Chapter 1 may very well be a review for many.

Chapter 2 deals exclusively with logic. The goal here is to present what is needed to get into proofs as quickly as possible. Much of the emphasis in Chapter 2 is on statements, implications and quantified statements, including a discussion of mixed quantifiers. Sets are introduced before logic so that the student's first encounter with mathematics here is a familiar one and because sets are needed to discuss quantified statements properly in Chapter 2.

The two proof techniques of direct proof and proof by contrapositive are introduced in **Chapter 3** in the familiar setting of even and odd integers. Proof by cases is discussed in this chapter as well as proofs of "if and only if" statements. **Chapter 4** continues this discussion in other settings, namely divisibility of integers, congruence, real numbers and sets.

The technique of proof by contradiction is introduced in **Chapter 5**. Since existence proofs and counterexamples have a connection with proof by contradiction, these also occur in Chapter 5. The topic of uniqueness (of an element with specified properties) is also addressed in Chapter 5.

Proof by mathematical induction occurs in **Chapter 6**. In addition to the Principle of Mathematical Induction and the Strong Principle of Mathematical Induction, this chapter includes proof by minimum counterexample.

Chapter 7 reviews all proof techniques (direct proof, proof by contrapositive, proof by contradiction and induction) introduced in Chapters 3–6. This chapter provides many examples to solidify the understanding of these techniques, emphasizing both how and when to use the techniques. Exercises in this chapter are distributed randomly with respect to the method of proof used.

The main goal of **Chapter 8** (Prove or Disprove) concerns the testing of statements of unknown truth value, where it is to be determined, with justification, whether a given statement is true or false. In addition to the challenge of determining whether a statement is true or false, such problems provide added practice with counterexamples and the various proof techniques. Testing statements is preceded in this chapter by a historical discussion of conjectures in mathematics and a review of quantifiers.

Chapter 9 deals with relations, especially equivalence relations. Many examples involving congruence are presented and the set of integers modulo n is described.

Chapter 10 involves functions, with emphasis on the properties of one-to-one (injective) and onto (subjective) functions. This gives rise to a discussion of bijective functions and inverses of functions. The well-defined property of functions is discussed in more detail in this chapter. In addition, there is a discussion of images and inverse images of sets with regard to functions as well as operations on functions, especially composition.

Chapter 11 deals with infinite sets and a discussion of cardinalities of sets. This chapter includes a historical discussion of infinite sets, beginning with Cantor and his fascination and difficulties with the Schröder–Bernstein Theorem, then proceeding to Zermelo and the Axiom of Choice, and ending with a proof of the Schröder–Bernstein Theorem.

All of the proof techniques are employed in **Chapter 12**, where numerous results in the area of number theory are introduced and proved.

Chapter 13 deals with proofs in the area of discrete mathematics called combinatorics. The primary goal of this chapter is to introduce the basic principles of counting such as multiplication, addition, pigeonhole and inclusion-exclusion. The concepts of permutations and combinations described here give rise to a wide variety of counting problems. In addition, Pascal triangles and the related binomial theorem are discussed. This chapter describes many proofs that occur in this area including many examples of how this subject can be used to solve a variety of problems.

Chapter 14 deals with proofs that occur in calculus. Because these proofs are quite different from those previously encountered but are often more predictable in nature, many illustrations are given that involve limits of sequences and functions and their connections with infinite series, continuity and differentiability.

The final **Chapter 15** deals with modern algebra, beginning with binary operations and moving into proofs that are encountered in the area of group theory.

It is our experience that many students have benefited by reading and solving problems in these later chapters that deal with courses they are currently taking or are about to take. The same is true for the following online chapters.

The study of proofs in modern algebra continues in the first online **Chapter 16** (`goo.gl/zKdQor`), where the major topic is ring theory. Proofs concerning integral domains and fields are presented here as well.

In **Chapter 17** (`goo.gl/Tmh2ZB`), we discuss proofs in linear algebra, where the concepts of vector spaces and linear transformations are emphasized.

Even though real numbers (and, to a lesser degree, complex numbers) occur throughout mathematics, there are many properties of these two classes of numbers of which students may be unaware. This is the topic for **Chapter 18** (`goo.gl/ymv5kJ`).

The final online chapter is **Chapter 19** (`goo.gl/bWFRMu`), where we discuss proofs in topology. Included in this chapter are proofs involving metric spaces and topological spaces. A major topic here is open and closed sets.

Exercises

There are over 1000 exercises in Chapters 1–19. The degree of difficulty of the exercises ranges from routine to medium difficulty to moderately challenging. As mentioned earlier, the fourth edition contains more exercises in the moderately difficult category. Types of exercises include:

- Exercises that present students with statements, asking students to decide whether they are true or false (with justification).
- Proposed proofs of statements, asking if the argument is valid.
- Proofs without a statement given, which ask students to supply a statement of what has been proved.
- Exercises that call upon students to make conjectures of their own and possibly to provide proofs of these conjectures.

Chapter 3 is the first chapter in which students will be called upon to write proofs. At such an early stage, we feel that students need to (1) concentrate on constructing a valid proof and not be distracted by unfamiliarity with the mathematics, (2) develop some self-confidence with this process and (3) learn how to write a proof properly. With this in mind, many of the exercises in Chapter 3 have been intentionally structured so as to be similar to the examples in that chapter.

Exercises for each section in Chapters 1–6 and 8–15 occur at the end of a section (section exercises) and additional exercises for the entire chapter (supplemental exercises) appear at the end of the chapter as do chapter exercises for Chapter 7. Answers or hints to the odd-numbered section exercises appear at the end of the text as do odd-numbered chapter exercises for Chapter 7. One should also keep in mind, however, that proofs of results are not unique in general.

Teaching a Course from This Text

Although a course using this textbook could be designed in many ways, here are our views on such a course. As we noted earlier, we think it is useful for students to reread (at least portions of) Chapter 0 throughout the course, as we feel that with each reading, the chapter becomes more meaningful. The first part of Chapter 1 (Sets) will likely be familiar to most students, although the last part may not. Chapters 2–6 will probably be part of any such course, although certain topics could receive varying degrees of emphasis (with perhaps proof by minimum counterexample in Chapter 6 possibly even omitted). Chapter 7 reviews all proof techniques introduced in Chapters 3–6. If the instructor believes that students have obtained a strong understanding of these techniques, this chapter could be omitted. Nevertheless, we feel it is good for students to read this chapter and try solving the exercises. Instructors who choose to omit Chapter 7 might find it useful to assign exercises from this chapter, asking students to determine which proof technique is to be used.

One could spend much or little time on Chapter 8, depending on how much time is used to discuss the large number of "prove or disprove" exercises. We think that most of Chapters 9 and 10 would be covered in such a course. It would be useful to cover some of the fundamental ideas addressed in Chapter 11 (Cardinalities of Sets). As time permits, portions of the later chapters could be covered, especially those of interest to the instructor, including the possibility of going to the web site for even more variety.

Supplements and Technology

Online Chapters

Four additional chapters, Chapters 16–19 (dealing with proofs in ring theory, linear algebra, real and complex numbers, and topology), can be found by going to: `goo.gl/bf2Nb3`.

Instructor's Solutions Manual (downloadable)

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The Instructor's Solutions Manual, written by the authors, provides worked-out solutions for all exercises in the text. It is available for download to qualified instructors from the Pearson Instructor Resource Center https://www.pearson.com/us/sign-in.html.

Chapter Presentations

This icon, found beside the supplemental exercises for each chapter, indicates a chapter presentation. The short URLs in the margin of the text provide students with direct access to the presentations in PDF form. The URL `goo.gl/bf2Nb3` provides access to the complete library of presentations in both PDF and editable LaTeX (Beamer) formats.

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Gary Chartrand
Albert D. Polimeni
Ping Zhang

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Communicating Mathematics

Quite likely, the mathematics you have already encountered consists of doing problems using a specific approach or procedure. These may include solving equations in algebra, simplifying algebraic expressions, verifying trigonometric identities, using certain rules to find and simplify the derivatives of functions, and setting up and evaluating a definite integral that gives the area of a region or the volume of a solid. Accomplishing all of these is often a matter of practice.

Many of the methods used to solve problems in mathematics are based on results in mathematics that were discovered by people and shown to be true. This kind of mathematics may very well be new to you and, as with anything that's new, there are things to be learned. But learning something new can be (and, in fact should be) fun. There are several steps involved here. The first step is discovering something in mathematics that we believe to be true. How does one discover new mathematics? This usually comes about by considering examples and observing that a pattern seems to be occurring with the examples. This may lead to a guess on our part as to what appears to be happening. We then have to convince ourselves that our guess is correct. In mathematics this involves constructing a proof showing what we believe to be true is, in fact, true. But this is not enough. We need to convince others that we are right. So we need to write a proof that is written so clearly and so logically that people who know the methods of mathematics will be convinced. Where mathematics differs from all other scholarly fields is that once a proof has been given of a certain mathematical statement, there is no longer any doubt. This statement is true. Period. There is no other alternative.

Our main emphasis here will be in learning how to construct mathematical proofs and learning to write proofs in such a manner that these proofs will be clear to and understood by others. Even though learning to guess new mathematics is important and can be fun, we will spend only a little time on this as it often requires an understanding of more mathematics than can be discussed at this time. But why would we want to discover new mathematics? While one possible answer is that it comes from the curiosity that most mathematicians possess, a more common explanation is that we have a problem to solve that requires knowing that some mathematical statement is true.

0.1 LEARNING MATHEMATICS

One of the major goals of this book is to assist you as you progress from an individual who uses mathematics to an individual who understands mathematics. Perhaps this will mark the beginning of you becoming someone who actually develops mathematics of your own. This is an attainable goal if you have the desire.

The fact that you've gone this far in your study of mathematics suggests that you have ability in mathematics. This is a real opportunity for you. Much of the mathematics that you will encounter in the future is based on what you are about to learn here. The better you learn the material and the mathematical thought process now, the more you will understand later. To be sure, any area of study is considerably more enjoyable when you understand it. But getting to that point will require effort on your part.

There are probably as many excuses for doing poorly in mathematics as there are strategies for doing well in mathematics. We have all heard students say (sometimes, remarkably, even with pride) that they are not good at mathematics. That's only an alibi. Mathematics can be learned like any other subject. Even some students who have done well in mathematics say that they are not good with proofs. This, too, is unacceptable. What is required is determination and effort. To have done well on an exam with little or no studying is nothing to be proud of. Confidence based on being well prepared is good, however.

Here is some advice that has worked for several students. First, it is important to understand what goes on in class each day. This means being present and being prepared for every class. After each class, recopy any lecture notes. When recopying the notes, express sentences in your own words and add details so that everything is as clear as possible. If you run into snags (and you will), talk them over with a classmate or your instructor. In fact, it's a good idea (at least in our opinion) to have someone with whom to discuss the material on a regular basis. Not only does it often clarify ideas, it gets you into the habit of using correct terminology and notation.

In addition to learning mathematics from your instructor, solidifying your understanding by careful note-taking and talking with classmates, your textbook is (or at least should be) an excellent source as well. Read your textbook carefully with pen (or pencil) and paper in hand. Make a serious effort to do every homework problem assigned and, eventually, be certain that you know how to solve them. If there are exercises in the textbook that have not been assigned, you might even try to solve these as well. Another good idea is to try to create your own problems. In fact, when studying for an exam, try creating your own exam. If you start doing this for all of your classes, you might be surprised at how good you become. Creativity is a major part of mathematics. Discovering mathematics not only contributes to your understanding of the subject but has the potential to contribute to mathematics itself. Creativity can come in all forms. The following quote is from the well-known writer J. K. Rowling (author of the *Harry Potter* novels).

> *Sometimes ideas just come to me. Other times I have to sweat and almost bleed to make ideas come. It's a mysterious process, but I hope I never find out exactly how it works.*

In her book *Defying Gravity: The Creative Career of Stephen Schwartz from Godspell to Wicked*, Carol de Giere writes a biography of Stephen Schwartz, one of the most successful composer-lyricists, in which she discusses not only creativity in music but how an idea can lead to something special and interesting and how creative people may have to deal with disappointment. Indeed, de Giere dedicates her book to the *creative spirit within each of us*. While Schwartz wrote the music for such famous shows as *Godspell* and *Wicked*, he discusses creativity head-on in the song "The Spark of Creation," which he wrote for the musical *Children of Eden*. In her book, de Giere writes:

> *In many ways, this song expresses the theme of Stephen Schwartz's life – the naturalness and importance of the creative urge within us. At the same time he created an anthem for artists.*

In mathematics our goal is to seek the truth. Finding answers to mathematical questions is important, but we cannot be satisfied with this alone. We must be certain that we are right and that our explanation for why we believe we are correct is convincing to others. The reasoning we use as we proceed from what we know to what we wish to show must be logical. It must make sense to others, not just to ourselves.

There is joint responsibility here. As writers, it is our responsibility to give an accurate, clear argument with enough details provided to allow the reader to understand what we have written and to be convinced. It is the reader's responsibility to know the basics of logic and to study the concepts involved so that a well-presented argument will be understood. Consequently, in mathematics writing is important, *very* important. Is it *really* important to write mathematics well? After all, isn't mathematics mainly equations and symbols? Not at all. It is not only important to write mathematics well, it is important to write well. You will be writing the rest of your life, at least reports, letters and email. Many people who never meet you will know you only by what you write and how you write.

Mathematics is a sufficiently complicated subject that we don't need vague, hazy and boring writing to add to it. A teacher has a very positive impression of a student who hands in well-written and well-organized assignments and examinations. You want people to enjoy reading what you've written. It is important to have a good reputation as a writer. It's part of being an educated person. Especially with the large number of email letters that so many of us write, it has become commonplace for writing to be more casual. Although all people would probably subscribe to this (since it is more efficient), we should know how to write well, formally and professionally, when the situation requires it.

You might think that considering how long you've been writing and that you're set in your ways, it will be very difficult to improve your writing. Not really. If you want to improve, you can and will. Even if you are a good writer, your writing can always be improved. Ordinarily, people don't think much about their writing. Often just thinking about your writing is the first step to writing better.

0.2 WHAT OTHERS HAVE SAID ABOUT WRITING

Many people who are well known in their areas of expertise have expressed their thoughts about writing. Here are quotes by some of these individuals.

Anything that helps communication is good. Anything that hurts it is bad.

I like words more than numbers, and I always did—conceptual more than computational.

Paul Halmos, mathematician

Writing is easy. All you have to do is cross out all the wrong words.

Mark Twain, author (*The Adventures of Hucklebery Finn*)

You don't write because you want to say something; you write because you've got something to say.

F. Scott Fitzgerald, author (*The Great Gatsby*)

Writing comes more easily if you have something to say.

Scholem Asch, author

Either write something worth reading or do something worth writing.

Benjamin Franklin, statesman, writer, inventor

What is written without effort is in general read without pleasure.

Samuel Johnson, writer

Easy reading is damned hard writing.

Nathaniel Hawthorne, novelist (*The Scarlet Letter*)

Everything that is written merely to please the author is worthless.

The last thing one knows when writing a book is what to put first.

I have made this letter longer because I lack the time to make it short.

Blaise Pascal, mathematician and physicist

The best way to become acquainted with a subject is to write a book about it.

Benjamin Disraeli, prime minister of England

In a very real sense, the writer writes in order to teach himself, to understand himself, to satisfy himself; the publishing of his ideas, though it brings gratification, is a curious anticlimax.

Alfred Kazin, literary critic

The skill of writing is to create a context in which other people can think.

Edwin Schlossberg, exhibit designer

A writer needs three things, experience, observation, and imagination, any two of which, at times any one of which, can supply the lack of the other.

William Faulkner, writer (*The Sound and the Fury*)

If confusion runs rampant in the passage just read,
It may very well be that too much has been said.

So that's what he meant! Then why didn't he say so?

Frank Harary, mathematician

A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.

David Hilbert, mathematician

Everything should be made as simple as possible, but not simpler.

Albert Einstein, physicist

Never let anything you write be published without having had others critique it.

Donald E. Knuth, computer scientist and writer

Some books are to be tasted, others to be swallowed, and some few to be chewed and digested.

Reading maketh a full man, conference a ready man, and writing an exact man.

Francis Bacon, writer and philosopher

Judge an article not by the quality of what is framed and hanging on the wall, but by the quality of what's in the wastebasket.

Anonymous (Quote by Leslie Lamport)

We are all apprentices in a craft where no-one ever becomes a master.

Ernest Hemingway, author (*For Whom the Bell Tolls*)

There are three rules for writing a novel. Unfortunately, no one knows what they are.

W. Somerset Maugham, author (*Of Human Bondage*)

0.3 MATHEMATICAL WRITING

Most of the quotes given above pertain to writing in general, not to mathematical writing in particular. However, these suggestions for writing apply as well to writing mathematics. For us, mathematical writing means writing assignments for a mathematics course (particularly a course with proofs). Such an assignment might consist of writing a single proof, writing solutions to a number of problems, or perhaps writing a term paper which, more than likely, includes definitions, examples, background *and* proofs. We'll refer to any of these as an "assignment." Your goal should be to write correctly, clearly and in an interesting manner.

Before you even begin to write, you should have already thought about a number of things. First, you should know what examples and proofs you plan to include if this is appropriate for your assignment. You should not be overly concerned about writing good proofs on your first attempt – but be certain that you do have *proofs*.

As you're writing your assignment, you must be aware of your audience. What is the target group for your assignment? Of course, it should be written for your instructor. But it should be written so that a classmate would understand it. As you grow mathematically, your audience will grow with you and you will adapt your writing to this new audience.

Give yourself enough time to write your assignment. Don't try to put it together just a few minutes before it's due. The disappointing result will be obvious to your instructor. And to you! Find a place to write that is comfortable for you: your room, an office, a study room, the library and sitting at a desk, at a table, in a chair. Do what works best for you. Perhaps you write best when it's quiet or when there is background music.

Now that you're comfortably settled and have allowed enough time to do a good job, let's put a plan together. If the assignment is fairly lengthy, you may need an outline, which, most likely, will include one or more of the following:

1. background and motivation
2. definitions to be presented and possibly notation to be used
3. examples to include
4. results to be presented (whose proofs have already been written, probably in rough form)
5. references to other results you intend to use
6. the order of everything mentioned above.

If the assignment is a term paper, it may include extensive background material and may need to be carefully motivated. The subject of the paper should be placed in perspective. Where does it fit in with what we already know?

Many writers write in "spirals." Even though you have a plan for your assignment that includes an ordered list of things you want to say, it is likely that you will reach some point (perhaps sooner than you think) when you realize that you should have included something earlier – perhaps a definition, a theorem, an example, some notation. (This happened to us many times while writing this textbook.) Insert the missing material, start over again and write until once again you realize that something is missing. It is important, as you reread, that you start at the beginning each time. Then repeat the steps listed above.

We are about to give you some advice, some "pointers," about writing mathematics. Such advice is necessarily subjective. Not everyone subscribes to these suggestions on writing. Indeed, writing "experts" don't agree on all issues. For the present, your instructor will be your best guide. But writing does not follow a list of rules. As you mature mathematically, perhaps the best advice about your writing is the same advice given by Jiminy Cricket to Disney's Pinocchio: *Always let your conscience be your guide.* You must be yourself. And one additional piece of advice: Be careful about accepting advice on writing. Originality and creativity don't follow rules. Until you reach the stage of being comfortable and confident with your own writing, however, we believe that it is useful to consider a few writing tips.

Since a number of these writing tips may not make sense (since, after all, we don't even have anything to write yet), it will probably be most useful to return to this chapter periodically as you proceed through the chapters that follow.

0.4 USING SYMBOLS

Since mathematics is a symbol-oriented subject, mathematical writing involves a mixture of words and symbols. Here are several guidelines to which a number of mathematicians subscribe.

1. *Never start a sentence with a symbol.*
   Writing mathematics follows the same practice as writing all sentences, namely that the first word should be capitalized. This is confusing if the

sentence were to begin with a symbol since the sentence appears to be incomplete. Also, in general, a sentence sounds better if it starts with a word. Instead of writing:

$$x^2 - 6x + 8 = 0 \text{ has two distinct roots.}$$

write:

$$\text{The equation } x^2 - 6x + 8 = 0 \text{ has two distinct roots.}$$

2. *Separate symbols not in a list by words if possible.*
Separating symbols by words makes the sentence easier to read and therefore easier to understand. The sentence:

With the exception of a, b is the only root of $(x - a)(x - b) = 0$.

would be clearer if it were written as:

With the exception of a, the number b is the only root of $(x - a)(x - b) = 0$.

3. *Except when discussing logic, avoid writing the following symbols in your assignment:*

$$\Rightarrow,\ \forall,\ \exists,\ \ni,\ etc.$$

The first four symbols stand for "implies," "for every," "there exists" and "such that," respectively. You may have already seen these symbols and know what they mean. If so, this is good. It is useful when taking notes or writing early drafts of an assignment to use shorthand symbols but many mathematicians avoid such symbols in their professional writing. (We will visit these symbols later.)

4. *Be careful about using i.e. and e.g.*
These stand for *that is* and *for example,* respectively. There are situations when writing the words is preferable to writing the abbreviations as there may be confusion with nearby symbols. For example, $\sqrt{-1}$ and $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ are not rational numbers, that is, i and e are not rational numbers.

5. *Write out integers as words when they are used as adjectives and when the numbers are relatively small or are easy to describe in words. Write out numbers numerically when they specify the value of something.*

There are exactly two groups of order 4.
Fifty million Frenchmen can't be wrong.
There are one million positive integers less than 1,000,001.

6. *Don't mix words and symbols improperly.*
Avoid writing:

Every integer ≥ 2 is a prime or is composite.

It is preferable to write:

Every integer exceeding 1 is prime or composite.

or

If $n \geq 2$ is an integer, then n is prime or composite.

Although

Since $(x-2)(x-3)=0$, it follows that $x=2$ or 3.

sounds correct, it is not written correctly. It should be:

Since $(x-2)(x-3)=0$, it follows that $x=2$ or $x=3$.

7. Avoid using a symbol in the statement of a theorem when it's not needed. Don't write:

Theorem *Every bijective function f has an inverse.*

Delete "f." It serves no useful purpose. The theorem does not depend on what the function is called. A symbol should not be used in the statement of a theorem (or in its proof) exactly once. If it is useful to have a name for an arbitrary bijective function in the proof (as it probably will be), then "f" can be introduced there.

8. *Explain the meaning of every symbol that you introduce*.
Although what you intended may seem clear, don't assume this. For example, if you write $n = 2k + 1$ and k has never appeared before, then say that k is an integer (if indeed k *is* an integer).

9. *Use "frozen symbols" properly*.
If m and n are typically used for integers (as they probably are), then don't use them for real numbers. If A and B are used for sets, then don't use these as typical elements of a set. If f is used for a function, then don't use this as an integer. Write symbols that the reader would expect. To do otherwise could very well confuse the reader.

10. *Use consistent symbols.*
Unless there is some special reason to the contrary, use symbols that "fit" together. Otherwise, it is distracting to the reader.
Instead of writing:

If x and y are even integers, then $x = 2a$ and $y = 2r$ for some integers a and r.

replace $2r$ by $2b$ (where then, of course, we write "for some integers a and b"). On the other hand, you might prefer to write $x = 2r$ and $y = 2s$.

0.5 WRITING MATHEMATICAL EXPRESSIONS

There will be numerous occasions when you will want to write mathematical expressions in your assignment, such as algebraic equations, inequalities and formulas. If these expressions are relatively short, then they should probably be written within the text of the proof or discussion. (We'll explain this in a moment.) If the expressions are rather lengthy, then it is probably preferred for these expressions to be written as "displays."

For example, suppose that we are discussing the Binomial Theorem. (It's not important if you don't recall what this theorem is.) It's possible that what we are writing includes the following passage:

> For example, if we expand $(a+b)^4$, then we obtain $(a+b)^4 = a^4 + 4a^3b +$
> $6a^2b^2 + 4ab^3 + b^4$.

It would probably be better to write the expansion of $(a+b)^4$ as a **display**, where the mathematical expression is placed on a line or lines by itself and is centered. This is illustrated below.

> For example, if we expand $(a+b)^4$, then we obtain
> $$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

If there are several mathematical expressions that are linked by equal signs and inequality symbols, then we would almost certainly write this as a display. For example, suppose that we wanted to write $n^3 + 3n^2 - n + 4$ in terms of k, where $n = 2k+1$. A possible display is given next:

> Since $n = 2k+1$, it follows that
> $$\begin{aligned} n^3 + 3n^2 - n + 4 &= (2k+1)^3 + 3(2k+1)^2 - (2k+1) + 4 \\ &= (8k^3 + 12k^2 + 6k + 1) + 3(4k^2 + 4k + 1) - 2k - 1 + 4 \\ &= 8k^3 + 24k^2 + 16k + 7 = 8k^3 + 24k^2 + 16k + 6 + 1 \\ &= 2(4k^3 + 12k^2 + 8k + 3) + 1. \end{aligned}$$

Notice how the equal signs are lined up. (We wrote two equal signs on one line since that line would have contained very little material otherwise, as well as to balance the lengths of the lines better.)

Let's return to the expression $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ for a moment. If we were to write this expression in the text of a paragraph (as we are doing) and if we find it necessary to write portions of this expression on two separate lines, then this expression should be broken so that the first line ends with an operation or comparative symbol such as $+$, $-$, $<$, $\geq$ or $=$. In other words, the second line should *not* begin with one of these symbols. The reason for doing this is that ending the line with one of these symbols alerts the reader that more will follow; otherwise, the reader might conclude (incorrectly) that the portion of the expression appearing on the first line is the entire expression. Consequently, write

> For example, if we expand $(a+b)^4$, then we obtain $(a+b)^4 = a^4 + 4a^3b +$
> $6a^2b^2 + 4ab^3 + b^4$.

and not

> For example, if we expand $(a+b)^4$, then we obtain $(a+b)^4 = a^4 + 4a^3b$
> $+\ 6a^2b^2 + 4ab^3 + b^4$.

If there is an occasion to refer to an expression that has already appeared, then this expression should have been written as a display and labeled as below:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4. \qquad \textbf{(1)}$$

Then we can simply refer to expression (1) rather then writing it out each time.